Journal of Engineering Mathematics, Vol. 8, No. 2, April 1974
Noordhoff International Publishing–Leyden
Printed in The Netherlands

Potential theory problem for two strips in contact

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(Received July 17, 1973)

SUMMARY
The problem of two geometrically identical strips in contact to form the potential theory equivalent to a lap joint is formulated and reduced, by integral transform methods, to the solution of a singular integral equation. A collocation scheme is used to obtain numerical results for various overlaps and amounts of coupling between the strips.

1. Introduction

An important problem in elastostatics is that of two strips that are bonded together in what is usually called a *lap joint*. The geometrical configuration for such a bonded structure is shown in Fig. 1, where the upper strip occupies the region, $-\pi \leqq y \leqq 0$, $-a \leqq x < \infty$, and the lower strip the region, $0 \leqq y \leqq \pi$, $-\infty < x \leqq a$. The two strips will usually be bonded in the region, $y=0$, $-a \leqq x \leqq a$, and if any peeling should occur, there may be debonding reducing the contact region to $-c \leqq x \leqq c$. It is not the intention to solve the elasticity problem here but rather to restrict the attention to the related potential theory problem.

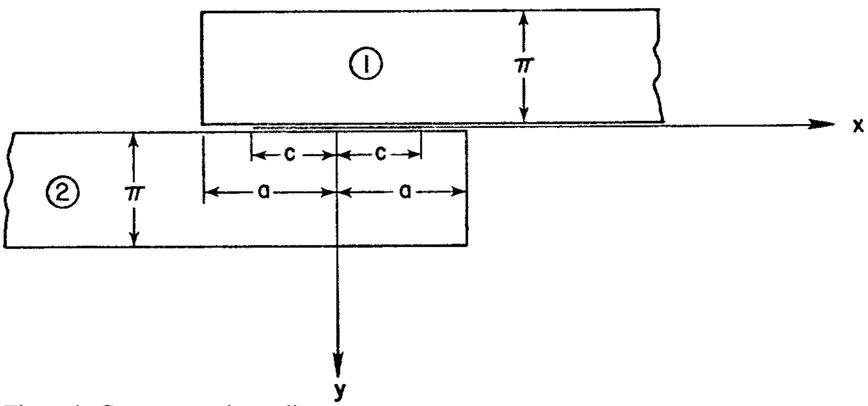

Figure 1. Geometry and coordinate system.

The potential theory problem requires the determination of two functions, ϕ_1 and ϕ_2, where subscript "1" refers to the upper strip and "2" to the lower one. The two functions satisfy the two dimensional Laplace equation and appropriate boundary and matching conditions. The boundary conditions are the following

$$\frac{\partial \phi_1}{\partial y} = 0 \qquad y = -\pi\,, \quad -a \leqq x < \infty \tag{1}$$

$$\frac{\partial \phi_1}{\partial x} = 0 \qquad x = -a\,, \quad -\pi \leqq y \leqq 0 \tag{2}$$

$$\frac{\partial \phi_1}{\partial y} = 0 \qquad y = 0\,, \quad -a < x < -c\,, \quad c < x < \infty \tag{3}$$

$$\frac{\partial \phi_2}{\partial y} = 0 \quad y = \pi, \quad -\infty < x \leqq a \tag{4}$$

$$\frac{\partial \phi_2}{\partial x} = 0 \quad x = a, \quad 0 \leqq y \leqq \pi \tag{5}$$

$$\frac{\partial \phi_2}{\partial y} = 0 \quad y = 0, \quad -\infty < x < -c, \; c < x < a. \tag{6}$$

In addition, the following matching conditions must be applied:

$$\frac{\partial \phi_1}{\partial y} = \frac{\partial \phi_2}{\partial y} \qquad y = 0, \; -c \leqq x \leqq c \tag{7}$$

$$\frac{\partial \phi_1}{\partial x} = \beta \frac{\partial \phi_2}{\partial x} \qquad y = 0, \; -c \leqq x \leqq c. \tag{8}$$

The physical problems posed are the following. If one assumes that ϕ_1 and ϕ_2 are equal to $k_1 T_1$ and $k_2 T_2$, where k_i and $T_i(x, c)$ $(i=1, 2)$ are the thermal conductivities and temperatures in the upper and lower strips, then the boundary conditions represent the steady state heat flow problem of two strips in contact having different thermal conductivities. Here $\beta = k_1/k_2$ and the strips are thermally insulated except in the region, $y=0$, $-c \leqq x \leqq c$. The continuity of heat flux from one strip to the other may be written as

$$\frac{\partial \phi_1}{\partial y} = \frac{\partial \phi_2}{\partial y} = a(x) H(c - |x|) \qquad y = 0 \tag{9}$$

where $H(x)$ is the Heaviside function. The total rate of heat flow across $y=0$ is then

$$\int_{-c}^{c} a(x)\,dx = 1. \tag{10}$$

In addition, the following conditions must hold as $x \to \pm\infty$:

$$\lim_{x\to\infty} \int_{-\pi}^{0} \frac{\partial \phi_1}{\partial x}\,dy = \lim_{x\to-\infty} \int_{0}^{\pi} \frac{\partial \phi_2}{\partial x}\,dy = -1 \tag{11}$$

which reflect the fact that there must be the same total heat flow leaving the bottom strip as entering the top one.

The problem could also be viewed as an elasticity problem where two semi-infinite layers are bonded together and put into a state of anti-plane shear. For that problem $\phi_i(x, y) = w_i(x, y)$, the displacements in the z-direction. The stresses are $\tau_{zx}^{(i)} = \partial\phi_i/\partial x$, $\tau_{zy}^{(i)} = \partial\phi_i/\partial y$ $(i=1, 2)$ and $\beta = \mu_2/\mu_1$ is the ratio of shear moduli. Equation (7) is the continuity of shear stresses through the contact region and Eqn. (8) is the condition of displacement continuity differentiated in the tangential direction. Equation (10) now gives the total load as an integration of the contact stresses. Equilibrium for each layer is satisfied by equating the sum of Eqns. (10) and (11) to zero.

2. Reduction to integral equation

The problem can be reduced to an integral equation by relatively simple means through the use of the exponential and finite Fourier transforms (see, e.g. Sneddon [1]). Convenient representations for the potential functions are the following:

$$\phi_1(x, y) = \int_{-\infty}^{\infty} \left[A_1(\xi) e^{|\xi| y} + C_1(\xi) e^{-|\xi| y}\right] e^{-i\xi x} d\xi + \tfrac{1}{2} B_0 x + \sum_{n=1}^{\infty} B_n e^{-n(a+x)} \cos(ny) \tag{12}$$

$$\phi_2(x, y) = \int_{-\infty}^{\infty} \left[A_2(\xi) e^{|\xi| y} + C_2(\xi) e^{-|\xi| y}\right] e^{-i\xi x} d\xi + \tfrac{1}{2} D_0 x + \sum_{n=1}^{\infty} D_n e^{-n(a-x)} \cos(ny). \tag{13}$$

From boundary conditions (1) and (4) the following relations are developed:

$$A_1(\xi)\,e^{-2|\xi|\pi} = C_1(\xi) \tag{14}$$

$$A_2(\xi) \qquad = C_2(\xi)\,e^{-2|\xi|\pi}. \tag{15}$$

Using Eqns. (9), (14), and (15) and the Fourier inversion theorem

$$2\pi|\xi|A_1(\xi)(1-e^{-2|\xi|\pi}) = \int_{-c}^{c} a(s)\,e^{i\xi s}\,ds \tag{16}$$

$$2\pi|\xi|C_2(\xi)(e^{-2|\xi|\pi}-1) = \int_{-c}^{c} a(s)\,e^{i\xi s}\,ds \tag{17}$$

where it is clear that Eqns. (3) and (6) are also automatically satisfied. Furthermore, from (7):

$$A_1(\xi) = -C_2(\xi), \quad C_1(\xi) = -A_2(\xi). \tag{18}$$

By application of finite Fourier transforms to boundary conditions (2) and (5), into which have been substituted Eqns. (12) and (13), and by summation of the resulting infinite series, the relatively simple representations for the Fourier coefficients given below are obtained:

$$nB_n = \frac{1}{\pi}\int_{-c}^{c} a(s)\,e^{-n(a+s)}\,ds \tag{19}$$

$$nD_n = -\frac{1}{\pi}\int_{-c}^{c} a(s)\,e^{-n(a-s)}\,ds \tag{20}$$

$$B_0 = \frac{1}{\pi}\int_{-c}^{c} a(s)ds \tag{21}$$

$$D_0 = -\frac{1}{\pi}\int_{-c}^{c} a(s)ds. \tag{22}$$

Boundary condition, Eqn. (8), with the results given in Eqns. (12) to (22) leads to the following singular integral equation:

$$\int_{-c}^{c} a(s)\left\{\frac{1}{s-x} - \left[\tfrac{1}{2}\,\mathrm{ctnh}\left(\frac{x-s}{2}\right) - \frac{1}{x-s}\right]\right.$$
$$\left. - \frac{1}{1+\beta}\left[\frac{\exp[-(2a+s+x)]}{1-\exp[-(2a+s+x)]} - \beta\,\frac{\exp[-(2a-s-x)]}{1-\exp[-(2a-s-x)]}\right]\right\}ds$$
$$= \frac{1}{2}\,\frac{1-\beta}{1+\beta} = \Gamma \qquad -c \leqq x \leqq c \tag{23}$$

which is to be solved in conjunction with Eqn. (10). It should be noted in Eqn. (23) that provided $c<a$ the last two terms in braces are regular for $-c\leqq s, x\leqq c$. However, if $c=a$ and $s, x\to\pm c$, then they can become singular. Such problems have been studied by the author and a colleague [2, 3, 4] and by Erdogan and Gupta [5]. The former group of problems were formulated so that the singularity arose in the kernel of a Fredholm integral equation while in the latter it arose from a singular integral equation with properties as above.

The problem can be solved most easily by the collocation technique of Erdogan and Gupta once the nature of the singularity has been deduced. The function $a(s)$ is found to have the following form:

$$a(s) = (c+s)^{\gamma-1}(c-s)^{\alpha-1}B(s) \tag{24}$$

where

$$\cos(\pi\alpha) = -\frac{\beta}{1+\beta}, \qquad \tfrac{1}{2}<\alpha<1 \tag{25}$$

$$\cos(\pi\gamma) = -\frac{1}{1+\beta}, \qquad \tfrac{1}{2}<\gamma<1 \tag{26}$$

and $B(s)$ is a regular function. Equations (25) and (26) which determine α and γ hold when $c=a$. When $c<a$, then α and γ are $\frac{1}{2}$. Equations (25) and (26) for $c=a$ are obtained in a rather elementary manner by consideration of Eqn. (23) in the vicinity of $x=\pm c$. For example, Eqn. (25) is obtained by considering the behavior of Eqn. (23) for $c=a$ as $s, x\to +c$. Assuming the unknown function, $a(s)$, to have the form

$$a(s)=(c-s)^{\alpha-1}A(s)\,. \tag{27}$$

Equation (23) can be written in a form involving a singular part and a regular part as

$$\int_{-c}^{c} a(s)\left[\frac{1}{s-x}+\frac{\beta}{1+\beta}\frac{1}{2c-s-x}\right]ds+R(x)=\frac{1}{2}\left(\frac{1-\beta}{1+\beta}\right) \tag{28}$$

where Eqn. (28) holds as $x\to +c$ and $R(x)$ is regular. Let $u=c-s$, and as $x\to +c$

$$A(c)\int_{0}^{2c}\left[\frac{-u^{\alpha-1}}{u-(c-x)}+\frac{\beta}{1+\beta}\frac{u^{\alpha-1}}{u+c-x}\right]du+R(x)=\frac{1}{2}\left(\frac{1-\beta}{1+\beta}\right). \tag{29}$$

The singular part of the integral may be extracted by considering the upper limit to be infinitely large, whence the integral is evaluated by its Mellin transform [6]. The result is

$$\pi A(c)(c-x)^{\alpha-1}\left[\operatorname{ctn}(\pi\alpha)+\frac{\beta}{1+\beta}\csc(\pi\alpha)\right]+R^{*}(x)=\frac{1}{2}\left(\frac{1-\beta}{1+\beta}\right). \tag{30}$$

If the singular term is to vanish, then Eqn. (25) must hold. A similar investigation as $x\to -c$ will produce Eqn. (26). The nature of the singularity could also have been determined by alternate techniques. For example, the right-angle corner could be investigated by using the method of Knien [7] and Williams [8] as an independent check. Such an investigation would reveal the same conclusions as those deduced from the integral equation.

3. Numerical analysis

The most suitable method to use for the numerical analysis is that exploited by Erdogan and Gupta [5], which is a scheme that directly integrates Eqn. (23) by means of the Gauss–Jacobi integration formula. Equation (23) is first normalized by letting $s=ct$, $x=c\eta$, and then the quadrature is performed by using the roots of the Jacobi polynomial

$$P_n^{(\alpha-1,\gamma-1)}(t_k)=0\,, \qquad k=1,2,\ldots,n\,. \tag{31}$$

The collocation points, η_r, are obtained as the roots of the Jacobi polynomial

$$P_{n-1}^{(\alpha,\gamma)}(\eta_r)=0\,, \qquad r=1,2,\ldots,n-1\,. \tag{32}$$

When $c<a$ then $\alpha=\gamma=\frac{1}{2}$ and the quadrature process is Gauss–Tchebichef [9]. When $c=a$, then α and γ are given by Eqns. (24) and (25).

Equations (23) and (10) become the following system of simultaneous algebraic equations:

$$\sum_{k=1}^{n} cA_kQ(t_k)\left\{\tfrac{1}{2}\coth\left[\tfrac{1}{2}c(t_k-\eta_r)\right]-\frac{1}{1+\beta}\left[\frac{\exp\left[-(2a+ct_k+c\eta_r)\right]}{1-\exp\left[-(2a+ct_k+c\eta_r)\right]}\right.\right.$$

$$\left.\left.-\beta\frac{\exp\left[-(2a-ct_k-c\eta_r)\right]}{1-\exp\left[-(2a-ct_k-c\eta_r)\right]}\right]\right\}=\frac{1}{2}\left(\frac{1-\beta}{1+\beta}\right) \qquad r=1,2,\ldots,n-1 \tag{33}$$

$$\sum_{k=1}^{n} cA_kQ(t_k)=1 \tag{34}$$

where

$$B(s)=B(ct)=c^{2-\alpha-\gamma}Q(t) \tag{35}$$

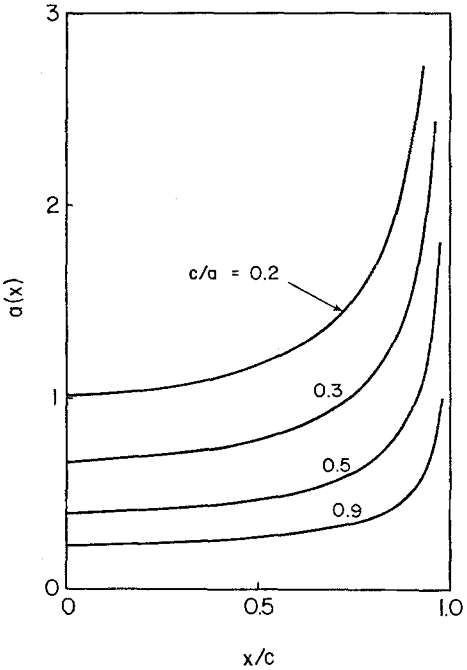

Figure 2. Flux distribution for $\Gamma=0$, $2a/\pi=1$ ($c<a$).

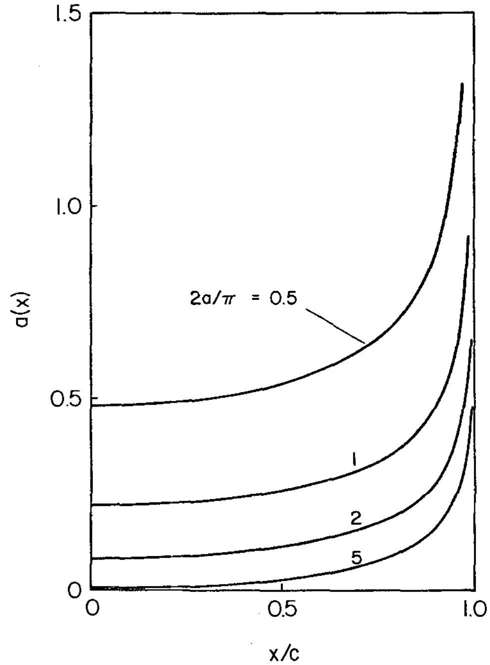

Figure 3. Flux distribution for $\Gamma=0$ ($c=a$).

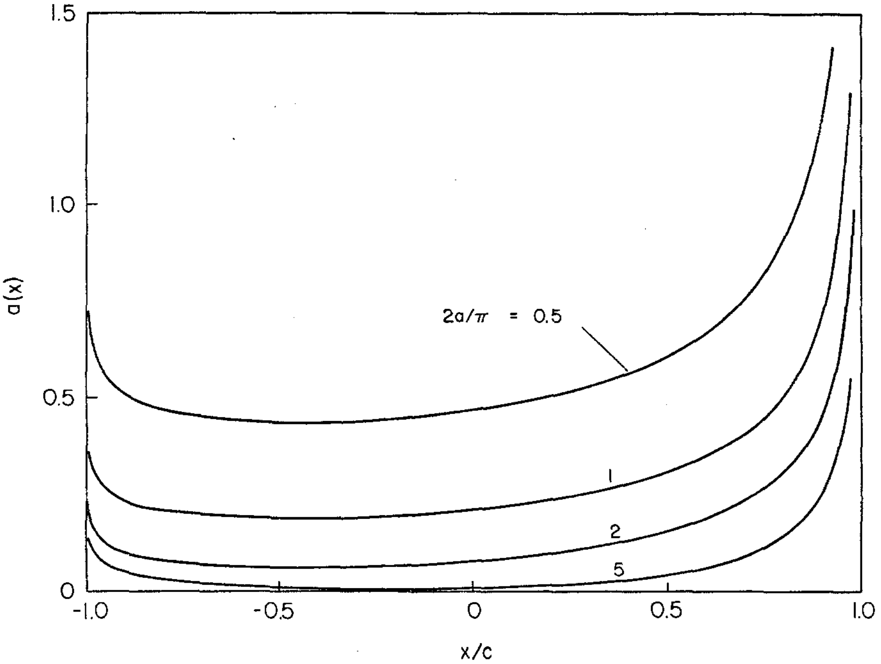

Figure 4. Flux distribution for $\Gamma=0.7$ ($c=a$).

and A_k are the weights of the Gauss–Jacobi integration formula.

Numerical results are given for several values of strip geometry and coupling constant. Figures 2 and 3 show curves of the flux/unit length along the contact region when $\beta=1$ (identical strips); Fig. 2 gives the case when $\alpha=\gamma=\frac{1}{2}$ ($c<a$) and Fig. 3 gives the case when $\alpha=\gamma=\frac{2}{3}$ ($c=a$). Figure 4 is for the particular case, $\beta=0.176471$ ($\Gamma=0.7$) and shows the antisymmetry of the flux distribution.

TABLE 1

Intensity factors for $c=a$

$2a/\pi$	$\Gamma=0, \alpha=0.667, \gamma=0.067$		$\Gamma=0.3, \alpha=0.614, \gamma=0.725$		$\Gamma=0.5, \alpha=0.580, \gamma=0.720$		$\Gamma=0.7, \alpha=0.547, \gamma=0.823$	
	$B(C)$	$B(-C)$	$B(C)$	$B(-C)$	$B(C)$	$B(-C)$	$B(C)$	$B(-C)$
0.1	0.645	0.645	0.669	0.616	0.685	0.594	0.704	0.572
0.5	0.401	0.401	0.417	0.374	0.421	0.349	0.420	0.317
1.0	0.366	0.366	0.385	0.331	0.389	0.296	0.385	0.250
2.0	0.390	0.390	0.413	0.341	0.415	0.291	0.405	0.222
3.0	0.432	0.432	0.452	0.379	0.449	0.320	0.431	0.236
5.0	0.507	0.507	0.517	0.452	0.503	0.385	0.471	0.283
7.0	0.565	0.565	0.567	0.512	0.543	0.439	0.499	0.323
10.0	0.634	0.634	0.624	0.583	0.590	0.503	0.532	0.372

TABLE 2

Intensity factors for $c<a$, $\Gamma=0$

$2a/\pi$ \ c/a	0.05	0.1	0.2	0.3	0.5	0.7	0.9
0.5	—	0.318	0.317	0.316	0.311	0.298	0.262
1.0	0.318	0.318	0.319	0.319	0.320	0.316	0.290
2.0	0.319	0.320	0.323	0.329	0.347	0.366	0.363
3.0	0.319	0.321	0.330	0.343	0.380	0.421	0.441
5.0	0.320	0.326	0.348	0.380	0.454	0.529	0.581

Of possible interest is the intensity of flux near the corners and this is given in Tables 1 and 2. In the tables the intensity is simply defined as $B(c)$ for $x=c$ and $B(-c)$ for $x=-c$. In the case of the lap joint in anti-plane shear one might modify the intensity factor by a constant to correspond to the designation used in fracture mechanics.

4. Discussion

The present method of analysis would appear suitable for a broader class of potential theory problems than treated here. It would, for example, require slight modification of the analysis to study strips in contact at right angles or at arbitrary angles. However, problems of plane elasticity, which involve the biharmonic equation, pose greater, although not insurmountable, difficulty.

5. Acknowledgement

The author is grateful for the support of the John Simon Guggenheim Memorial Foundation during his stay at the Department of Mathematics, University of Glasgow, where the research was completed. The author is grateful to Mr. Kasem Chantaramungkorn for performing the numerical work given in this paper. This research was supported in part by the National Foundation.

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